Equations, Lenses and Fractions

The study of lenses offers a good real world example of a relation with fractions we just can’t avoid! Different uses of a simple lens that you may be familiar with are using a magnifying glass to read small print, focusing a camera lens to take a picture, and using an overhead projector.

The focal length of a lens is the distance from the lens at which parallel light rays focus. It is easy to find the focal length for a magnifying glass by holding the lens out in bright sunlight and moving it to and from a surface until you see one small point of light. The focal length is the distance from the lens to the point of light, as seen in the following figure below. All you need to do is measure that length. But be careful. The point of light will be quite hot and can burn you. It can set paper or wood on fire.

![Diagram of magnifying glass with focal length](image)

A lens can focus an object on one side of the lens onto a surface on the other side of the lens. This is what a camera and an overhead projector do. Note in the following figure that the image is upside down. An overhead projector uses a mirror to flip the image right-side up. In the following figure, we let $d$ represent the distance of the object from the lens and let $i$ represent the distance from the lens to a surface on which the image is projected.

![Diagram of camera with focal length](image)

For the image to be in focus, $d$, $i$, and $f$ must satisfy the equation \[
\frac{1}{f} = \frac{1}{d} + \frac{1}{i}.\] Your professor adjusts an overhead projector by moving the lens closer to or further from the object (the transparency) until the distances satisfy this equation. You adjust a camera by moving the lens closer to and further away from the film until $i$, the distance of the lens from the film, satisfies this equation.
Focal lengths for camera lenses are usually 20 mm to 70 mm. Low focal lengths are good for taking pictures of large landscapes. Larger focal lengths are good for close-up pictures. The lens of a disposable camera usually has a focal length of about 30 mm.

Suppose you have a camera with a 30 mm lens and you want to take a picture of a flower that is about 600 mm from you (almost 2 feet). To find the distance the lens needs to be from the film, you can solve the equation \( \frac{1}{30} = \frac{1}{600} + \frac{1}{i} \) for \( i \).

1. Where did the 30 and the 600 come from? How do we know each one is in the correct position in the equation?

2. Approximate the solution to the equation \( \frac{1}{30} = \frac{1}{600} + \frac{1}{i} \) numerically on your TI-83. Write the equations exactly the way you would input them in a calculator:

   \[ Y_1 = \] ______________

   \[ Y_2 = \] ______________

Now use the TABLE feature of your calculator to approximate the solution to the nearest 10 mm. Explain how you do this and why it works.

Tell why it is clear from the equation that the solution to the nearest 10 mm is not the exact solution.

Use TABLE to find the solution to the nearest mm. Then to the nearest tenth of a mm.

3. Graph \( Y_1 \) and \( Y_2 \) on your calculator. Observing the y-values in the Table can help you set Ymin and Ymax in WINDOW. How can you use the graphs to solve the equation?

Explain why what you described gives the solution to the equation.

From your graph, solve \( \frac{1}{30} = \frac{1}{600} + \frac{1}{i} \), accurate to 4 decimal places.

4. Next we study the algebraic approach. The “difficulty” with this equation is that it has fractions in it. We wish it didn’t! We can make our wish come true. Consider that a fraction is just a division problem. \( \frac{x}{2} \) means “x divided by 2.” \( \frac{1}{i} \) means “1 divided by i.”

What operation un-does division? (What is the opposite of division?) Consider as an example the equation \( \frac{1}{3} = 5 + \frac{1}{x} \). It has two fractions, so it has two “divisors” (numbers that are doing the dividing). We could multiply both sides of this equation by 3x.
\[3x \cdot \frac{1}{3} = 3x \left( 5 + \frac{1}{x} \right).\] Why is this a good idea?

Now we have \( x = 15x + 3. \) Do you see what happened?
5. In solving \( \frac{1}{30} = \frac{1}{600} + \frac{1}{i} \), what should you multiply both sides by?

Solve \( \frac{1}{30} = \frac{1}{600} + \frac{1}{i} \), giving your answer as a fraction, reduced to lowest terms, and then as a decimal to 5 decimal places.

6. Look at a camera and estimate how far, in mm, the film is from the lens. (Look at a metric ruler if you need help with this estimate.) Is your solution to the equation in question 5 consistent with your estimated distance?

7. Measure the distance of a transparency from the lens of the overhead projector. Then measure how far the lens is from the screen. (Or, your professor may measure these for you.) Use these values to approximate \( f \), the focal length of the lens on the overhead projector.

8. Suppose you have a 30 mm lens on your camera and you want to take a close-up picture of a small shell. You try holding your camera about 2 inches (51 mm) from the shell.
   a. For this shot to be in focus, how far will the lens have to be from the film? What problem do you see with this?

   b. Suppose you hold your camera 1 inch (about 25 mm) from the shell. How far should the lens be from the film? Does this make sense?

9. Did you enjoy algebraically solving that equation with fractions in it in question 4? (Is the sky yellow? Do pigs fly?) But did you notice that the very first step you took was designed to get rid of all the fractions. That is always a good plan for solving an equation that has fractions. Remember:
   a) fractions are division expressions, and
   b) to un-do division we multiply.

To get an equation equivalent to this equation, \( \frac{1}{2} + \frac{9}{x} = \frac{5}{2x} \), you would want to multiply by one expression that would eliminate all three of those little divisors at once. What would work? _____ Go ahead and multiply every term of the equation by it.

Did you get an equation without fractions?
Then solve that.

Test your solution in \( \frac{1}{2} + \frac{9}{x} = \frac{5}{2x} \).
Does it satisfy the equation?
It should. (Re-check everything if it doesn't.)
10. Solve \( \frac{1}{2} + \frac{9}{x} = \frac{5}{2x} \) numerically. Explain what you do and why it works.

What equations do you put into your calculator?
What do you do then?
In what column of the table do you find the solution?
What is the solution?

11. Solve \( \frac{1}{2} + \frac{9}{x} = \frac{5}{2x} \) graphically. Explain what you do and why it works.

What equations do you put into your calculator?
What do you do then?
What window did you use? How did you decide what window would be useful?
What is your result?

12. Research question. The distance from the lens of the eye to the back of the retina would give us a slight underestimate of the focal length of the eye. See if you can find out what that distance is, on average. Given the focal length of the eye and the problem we saw in question 8, it might seem that we would be unable to see anything that was up close to our eye. Find out why it is that most of us can see things that are closer than the focal length of the average eye suggests.
Teaching Guide for
Equations, Lenses and Fractions

Introduction: The mathematical content of this lesson is the solution of equations that involve fractions. The goal is to focus the student’s attention on the meaning of solution of an equation more than to teach techniques for algebraic manipulation. Thus, the lesson includes solution of the equations by graphical, numerical, and algebraic methods. Using all three permits the student to focus on what the equation means and what it means to “solve” it.

Take a magnifying glass to class if you can. You can then show students how to find the focal length. Take a camera to class, too, and be sure that an overhead projector will be on hand.

Remark for Teachers Only: To derive the relationship among \( i \), \( d \), and \( f \) for a lens, observe the figure below, assuming that lines that look parallel are parallel. From the figure, you get, by similar triangles, \( \frac{d}{a} = \frac{i}{b} \) and \( \frac{i - f}{b} = \frac{f}{a} \). Solve for \( a \) in the first equation and substitute into the second equation. The \( b \)'s cancel out. The resulting equation can be rewritten as \( \frac{1}{f} = \frac{1}{d} + \frac{1}{i} \).

You shouldn’t present this to your class, but it is here for your information and to share with any student who inquires.

Answers and teaching suggestions:
1. The 30 and the 60 come from the information in the paragraph before the question. We have a lens with a 30 mm focal length, \( f \). The flower we want to photograph is at a distance of 600 mm, \( d \).

2. \( Y_1 = \frac{1}{30} \)
   \( Y_2 = \frac{1}{600} + \frac{1}{x} \)

   Now use the TABLE feature of your calculator to approximate the solution to the nearest 10 mm. Explain how you do this and why it works. Set TableStart at 0 (even though that asks for division by 0) or any multiple of 10. Set \( \Delta \text{Table} \) to 10. Then go to the table and look for the row where the two values for \( Y_1 \) and \( Y_2 \) differ the least. We’ll see

   \[
   \begin{array}{|c|c|c|}
   \hline
   X & Y_1 & Y_2 \\
   \hline
   20 & 0.03333 & 0.05167 \\
   \hline
   \end{array}
   \]
From this we estimate that the solution to the equation is \( i = 30 \), to the nearest 10. This is because 30 is the value of \( x=i \) that makes the value of \( \frac{1}{30} \) and the value of \( \frac{1}{600} + \frac{1}{i} \) most nearly equal. However, the observant student will note that this gives us
\[
\frac{1}{30} = \frac{1}{600} + \frac{1}{30},
\]
is clearly not true. It is useful to discuss this “equation.” It isn’t a true statement because we didn’t find the exact value of the solution, but it is approximately so; indeed, we know that the difference between the left side and the right side is exactly \( \frac{1}{600} \).

Use TABLE to find the solution to the nearest mm. Then to the nearest tenth of a mm. Looking at the table again, we see that the value \( X \) that would make \( Y_1 = Y_2 \) appears to be between 30 and 40 since the \( Y_2 \) values are decreasing and the value for \( Y_1 \) is greater than the value \( Y_2 \) when \( X=30 \), but \( Y_1 \) is less than \( Y_2 \) when \( X=40 \). So we reset the table with

\[
\begin{align*}
\text{TblStart} &= 30 \\
\Delta \text{Tbl} &= 1
\end{align*}
\]

This time we see that the solution is between 31 and 32.

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>.03333</td>
<td>.035</td>
</tr>
<tr>
<td>31</td>
<td>.03333</td>
<td>.03392</td>
</tr>
<tr>
<td>32</td>
<td>.03333</td>
<td>.03292</td>
</tr>
</tbody>
</table>

Thus, we revise the Table Setup once again, using

\[
\begin{align*}
\text{TblStart} &= 31 \\
\Delta \text{Tbl} &= 0.1
\end{align*}
\]

We see

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>31.5</td>
<td>.03333</td>
<td>.03341</td>
</tr>
<tr>
<td>31.6</td>
<td>.03333</td>
<td>.03331</td>
</tr>
<tr>
<td>31.7</td>
<td>.03333</td>
<td>.03321</td>
</tr>
</tbody>
</table>

So we conclude that the solution to the equation is between 31.5 and 31.6. These results are the point of the table search, but we can also answer the question that was actually asked. It is now clear that the solution to the nearest whole number was 32. However, to avoid obscuring the points we are making about what it means to solve an equation, either 31 or 32 should be accepted as estimates. We may feel confident that 31.6 is the solution to the nearest tenth, and it is acceptable to just say so. To test that with the table we would use

\[
\begin{align*}
\text{TableStart} &= 31.5, \ \Delta \text{Tbl} = .01
\end{align*}
\]

This would show us that \( x=31.58 \) gives both \( Y_1 = .03333 \) and \( Y_2 = .03333 \), assuring us that the solution is 31.6 to the nearest tenth.

3. Graph \( Y_1 \) and \( Y_2 \) on your calculator. Students have observed from the table that we need to be able to see \( X \)-values above 30, and that the \( Y \) values are very small and positive. They should use this to decide on a useful window. They might, for example, set

\[
\begin{align*}
\text{Xmin} &= 0 \\
\text{Xmax} &= 50
\end{align*}
\]
How can you use the graphs to solve the equation? The solution is the x-coordinate of the point of intersection. We can see that by a TRACE, ZOOM, TRACE, ZOOM process; or we can take advantage of the calculator’s CALC menu. Hitting 2nd CALC 5, and then Enter, Enter, Enter (to answer the calculator’s questions as Yes, Yes, Okay) will get us Intersection X=31.578947, Y=0.033333333.

Students should explain that this means that at the point on the graph where X=31.5789 (to 4 decimal places), both 1/30 and 1/600 + 1/x are equal to (about) 0.033333333. Thus, the solution to the equation is X=31.5789, to 4 decimal places.

4. Discussion of the equation $\frac{1}{3} = 5 + \frac{1}{x}$ is given to provide an example and help the students develop insight so that they can solve our lens equation independently. It is not necessary to complete the solution of $\frac{1}{3} = 5 + \frac{1}{x}$ beyond where we take it here. Be sure students discuss and understand the concepts that (1) fractions are division operations and (2) the inverse of division is multiplication, so that they understand that they can “un-do” division by multiplying by the divisors. Be sure, too, that they discuss and understand that if two equal expressions are to remain “equal,” then we will need to multiply both of them by the same amount. The problem tells students we will multiply both sides of this equation by 3x.

$$3x \cdot \frac{1}{3} = 3x \left(5 + \frac{1}{x}\right).$$

Why is this a good idea? Students should tell you that multiplying by 3 un-does the division by 3 in the fraction $\frac{1}{3}$, and multiplying by x un-does the division by x in the fraction $\frac{1}{x}$.

Now we have $x = 15x + 3$. Do you see what happened? With both of those divisions un-done, we have no fractions left in the equation and we can easily solve it.

5. In solving $\frac{1}{30} = \frac{1}{600} + \frac{1}{i}$, what should you multiply both sides by?

Solve $\frac{1}{30} = \frac{1}{600} + \frac{1}{i}$, giving your answer as a fraction, reduced to lowest terms, and then as a decimal to 5 decimal places.

In solving $1/30=1/600+1/i$ algebraically, some students may multiply both sides of the equation by $(30)(600)i$. After students have solved this equation, discuss the method of solution with the class. See if anyone realized that it could be solved by multiplying by 600i. Letting them struggle first with a larger number will help them appreciate the value of using the least common denominator. It is important that they realize they don’t NEED to use the least common denominator, only that it makes computations easier. To 5 decimal places, $i=31.57895$.

6. Look at a camera and estimate how far, in mm, the film is from the lens. (Look at a metric ruler if you need help with this estimate.) Is your solution to the equation in question 5 consistent with your estimated distance? Students really should see the camera and consider the actual distances so that they know this problem models the world they live in. Our solution,
$i = 31.6$, is seen to be reasonable; the distance from the lens to the film could certainly be 31.6 mm. Most students will need to hold a metric ruler as well as the camera to make this connection.

7. The equation and the solutions will depend on your overhead projector, how it is set up, and how accurately you are able to measure $d$ and $i$. Most often, an overhead projector designed for use in a small room has focal length of 292 mm, and an overhead projector designed for use in a large room has focal length of 355 mm. It is difficult to get accurate measurements, so you should be satisfied with any results in the ballpark.

8. Suppose you have a 30 mm lens on your camera and you want to take a close-up picture of a small shell. You try holding your camera about 2 inches (51 mm) from the shell.

   a. For this shot to be in focus, how far will the lens have to be from the film? What problem do you see with this? Students should write and solve the equation $\frac{1}{30} = \frac{1}{51} + \frac{1}{i}$.

      Encourage groups to assign individual members to use different methods (numerical, graphical, or algebraic). The solution is approximately $i = 73$ mm. Students will need to conclude that to focus the camera for this situation the lens needs to be 73 mm from the film. If you take a camera and a metric ruler to class, students will be able to see that this would not be possible; thus, such a shot could not be in focus.

   b. Suppose you hold your camera 1 inch (about 25 mm) from the shell. How far should the lens be from the film? Now they need to solve $\frac{1}{30} = \frac{1}{25} + \frac{1}{i}$. They will get $i \approx -150$. Maybe this means that the lens would have to be 150 mm behind the film; whatever it means, it does not make sense in this situation. We definitely will not be able to focus the shot.

9. Questions 9 and 10 are just practice, an opportunity to review and summarize the lesson. To get an equation equivalent to this equation $\frac{1}{2} + \frac{9}{x} = \frac{5}{2x}$, you would want to multiply $2x$ or $4x$.

   But students may want to try using $4x^2$. Doing so leads to a quadratic equation (which they may not yet know how to deal with) and one that leads to a solution candidate that will not satisfy the initial equation. This will help students realize that although the multiplier for the equations does not have to be the least common multiple of the denominators, it is very helpful if it contains the least common multiple of the variables.

   Did you get an equation without fractions? Proper multiplication by any common multiple of the denominators will produce an equation without fractions.

   Then solve that. The solution is $x = -13$.

   Test your solution in $\frac{1}{2} + \frac{9}{x} = \frac{5}{2x}$. We are again reminding students what it means to “solve” an equation. They should write and check that $\frac{1}{2} + \frac{9}{-13} = \frac{5}{2(-13)}$.
Checking, besides (1) reinforcing the meaning of a “solution” to an equation, also (2) reviews addition of fractions and (3) gives the student insight into what was going on in the algebra we did to solve.

\[
\frac{1}{2} + \frac{9}{-13} = \frac{5}{2(-13)} \Rightarrow \frac{1 \cdot (-13)}{2 \cdot (-13)} + \frac{9 \cdot (2)}{2(-13)} = \frac{5}{2(-13)}
\]

\[
\Rightarrow \frac{-13 + 18}{-26} = \frac{5}{-26}
\]

Does it satisfy the equation? Yes, we can see that it does.

10. Questions 10 and 11 each offer information in support of the other. It is important that students see and discuss this interaction of the numerical and graphical methods.

Solve \(\frac{1}{2} + \frac{9}{x} = \frac{5}{2x}\) numerically. Explain what you do and why it works.

What equations do you put into your calculator? \(Y_1 = \frac{1}{2} + 9/x\) and \(Y_2 = 5/(2x)\). This is an opportunity for students to discover or for you to tell them that some calculators will read \(5/2x\) as \((5/2)x\). Since the denominator of the fraction is \(2x\), they will need parentheses around \(2x\).

What do you do then? Set the table. For example: TableStart = 0 and ΔTable = 10. When we go to the table, we see that the values of \(Y_1\) and \(Y_2\) do not seem to get close together as \(x\) increases. Students may want to look at the graph at this point to see what is going on. Or, they may immediately think of going in the other direction to look at negative values of \(x\) in the table. Doing so, they see

<table>
<thead>
<tr>
<th>(X)</th>
<th>(Y_1)</th>
<th>(Y_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20</td>
<td>.05</td>
<td>-.125</td>
</tr>
<tr>
<td>-10</td>
<td>-.4</td>
<td>-.25</td>
</tr>
<tr>
<td>0</td>
<td>ERROR</td>
<td>ERROR</td>
</tr>
</tbody>
</table>

This suggests that there could be a solution between -20 and -10 since \(Y_1 > Y_2\) when \(x = -20\) but \(Y_1 < Y_2\) when \(x = -10\).

So, we revise the Table Setup. Perhaps TableStart = -20 and ΔTable = 1. (TableStart could be either -10 or -20 and ΔTable could be either +1 or -1 since we can move either up or down in the table.) Exploring the table, we find

<table>
<thead>
<tr>
<th>(X)</th>
<th>(Y_1)</th>
<th>(Y_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-13</td>
<td>-.1923</td>
<td>-.1923</td>
</tr>
</tbody>
</table>

In what column of the table do you find the solution? The solution is in the \(X\) column.

What is the solution? \(X = -13\).

11. Solve \(\frac{1}{2} + \frac{9}{x} = \frac{5}{2x}\) graphically. Explain what you do and why it works.

What equations do you put into your calculator? \(Y_1 = \frac{1}{2} + 9/x\) and \(Y_2 = 5/(2x)\), as in question 10. What do you do then? Set the window in some useful way. Then find a point where the two graphs intersect.
What window did you use? How did you decide what window would be useful? Table values help to reduce the guesswork in this process. One window that would work is

\[
\begin{align*}
X_{\text{min}} &= -40 \\
X_{\text{max}} &= 40 \\
Y_{\text{min}} &= -1 \\
Y_{\text{max}} &= 1
\end{align*}
\]

What is your result? The graphs intersect at (-13, -0.1923); thus the solution to the equation is \( x = -13 \).

Observing the graph, students know they have solved an equation where the expressions were not linear. It is worth pointing out to them that the numerical and graphical methods that they have learned to use are powerful tools that permit them to solve many equations beyond the scope of their algebraic knowledge and that they are empowered to use these methods.