In the Herb Business, Part III
Factoring and Quadratic Equations

In the herbal medicine business, you and your partner sold 120 bottles of your best herbal medicine each week when you sold at your original price. You have found that for each dollar you increase the price per bottle, you sell 5 fewer bottles per week.

Let \( x \) represent the number of dollars you increase the price. Let \( s \) represent the number of bottles you sell. The relationship between number of bottles sold and the price increase is given by the equation \( s = 120 - 5x \). Do you see why? This equation gives us the following table:

<table>
<thead>
<tr>
<th>Dollars increase, ( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottles per week, ( s )</td>
<td>130</td>
<td>125</td>
<td>120</td>
<td>115</td>
<td>110</td>
<td>105</td>
</tr>
</tbody>
</table>

Suppose that with no price increase (\( x = 0 \)), you make a profit of $3 per bottle. If you raise the price by $1, your profit will be $4 per bottle. If you raise the price by $2, you make $5 per bottle. Let \( p \) represent your profit per bottle. Then the profit is given by the linear function \( p = 3 + x \). (Be sure you see why.) This gives the values in the following table.

<table>
<thead>
<tr>
<th>Dollars increase, ( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit per bottle, ( p )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

Your main concern is how much profit you will make each week given different possible price increases. Let \( y \) represent your profit each week.

1. How much profit do you now make each week (with no price change)? How much profit will you make each week if you increase the price by \( x = 1 \) dollar?

2. Complete the following table.

<table>
<thead>
<tr>
<th>Dollars increase, ( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit per week, ( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. To determine your weekly profit, \( y \), you multiply the number of bottles sold by the profit per bottle, \( y = ps \). Substitution of \( 120 - 5x \) for \( s \) and \( 3 + x \) for \( p \) gives \( y = (3 + x)(120 - 5x) \) as the equation for profit \( y \) in terms of dollars increase \( x \). Make sure this is clear to you; it is important! Then substitute 1 for \( x \) in this equation to see if you get the same profit that you found in problem 1.

4. Suppose you reduce the price per bottle by 3 dollars, that is, \( x = -3 \). Use the profit equation, to find your weekly profit. Why does this make sense? (Hint: Think in terms of your profit per bottle.)
5. Suppose you increase the price per bottle by 24 dollars, that is, \( x = 24 \). By the profit equation, what is your weekly profit? Why does this make sense? (Hint: Think in terms of number of bottles sold.)

6. You are interested in determining the price increase that will result in your maximum weekly profit. To investigate this, input the equation in your calculator as \( y_1 = (x + 3)(120 - 5x) \). Then look at your table of values and observe your weekly profit for various possible price increases. By how much will you increase the price? Why?

7. We will now graph the profit equation. Use WINDOW values \( \text{Xmin} = -5 \) and \( \text{Xmax} = 30 \). Observe the \( y \)-values in your table to help you pick an appropriate range; then set your values for \( \text{Ymin} \) and \( \text{Ymax} \). Graph your function.
   a. Is the graph a straight line?
   b. Where are the \( x \)-intercepts?
   c. Why does this make sense? (Think about your answers to questions 4 and 5.)
   d. Use this graph to determine how much you will increase the price. How does the “best” price increase compare to the price increases that result in 0 profit?

\[ y = (x + 3)(120 - 5x) \] is called a quadratic function. Often quadratic functions arise as the product of two linear functions, as in this case. \( x + 3 \) and \( 120 - 5x \) are called factors of the expression \( (x + 3)(120 - 5x) \). This expression is said to be written in factored form.

8. You have learned previously how to multiply two binomial expressions together using the distributive property. If you have forgotten, find it in your text and review. Rewrite \( (x + 3)(120 - 5x) \) by multiplying the terms together. If you write your answer this way, \[ y = -5x^2 + 105x + 360 \], we say that the quadratic is in standard form. Put this expression into your calculator. Compare the \( y \)-values of both forms of this expression using the TABLE feature of your calculator. The values should be the same. Notice the symmetry of this graph. Draw a line on the graph that would act as a mirror for the curve so that the reflection of the graph on one side of the mirror line would be the other half of the curve.
9. How does the factored form, \( y = (x + 3)(120 - 5x) \), inform us of what we will see on the graph?
Teaching Guide for
In the Herb Business, Part III
Factoring and Quadratic Equations

Introduction: The mathematics in this lesson provides foundations for study of quadratic functions and an introduction to the solution of quadratic equations. The focus is on a quadratic as a product of two linear factors and the zeros of a quadratic function. Consideration of the turning point of a quadratic function is also included.

Our approach to quadratics is unconventional. We look at quadratics in factored form first, investing each factor with contextual meaning. Students graph these quadratics and notice the connection between the x-intercepts and the solution to the equation where the quadratic is equal to 0. With this in mind, students will be able to use a graphing calculator to help them factor, while developing a good understanding of the connection between graphs and solutions to equations.

Quadratics may connect better with students’ prior knowledge if they realize that they often arise in practice from the product of 2 linear functions. That is the major focus of the approach to quadratics in these lessons.

Answers and teaching suggestions:
In developing the linear equation s=120-5x, some students may not recognize it as the line s = -5x+120. The form s = -5x+120 is the standard slope/intercept form. The form s=120-5x describes how the equation was developed, that is, the number of bottles sold was initially 120 and we are now subtracting 5x bottles from that.

A similar comment might be made about the equation p=3+x compared to p=x+3.

1. How much profit do you now make each week (with no price change)? 120 x 3 =$360
   How much profit will you make each week if you increase the price by x=1 dollar? 115 x 4 = $460. It is important for students to realize that they are multiplying (bottles/week) x (profit/bottle) to get profit/week.

2. Complete the following table.

<table>
<thead>
<tr>
<th>Dollars increase, x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit per week, y</td>
<td>130</td>
<td>250</td>
<td>360</td>
<td>460</td>
<td>550</td>
<td>630</td>
</tr>
</tbody>
</table>

   Students can see these results as the product of the corresponding results in the previous tables.

3. To determine your **weekly profit**, y, you multiply the number of bottles sold by the profit per bottle, y=ps. Substitution of 120 – 5x for s and 3 + x for p gives \( y = (3 + x)(120 - 5x) \) as the equation for profit y in terms of dollars increase x. **Make sure this is clear to you; it is important!** Then substitute 1 for x in this equation to see if you get the same profit that you found in problem 1.
In question 1, we were multiplying \((3+x)(120-5x)\) first evaluated at \(x=0\) and then at \(x = 1\). This should be clear at this stage of the lesson.

4. Suppose you reduce the price per bottle by 3 dollars, that is, \(x = -3\). Use the profit equation, to find your weekly profit? Why does this make sense? (Hint: Think in terms of your profit per bottle.)

The profit will be 0. Since the original profit/bottle was $3, a reduction in price of $3 wipes out the profit. Be sure students observe that \(y(0) = 0 \cdot 135 = 0\). 135 bottles at 0 profit on each bottle produces a product of 0.

5. Suppose you increase the price per bottle by 24 dollars, that is, \(x=24\). By the profit equation, what is your weekly profit? Why does this make sense? (Hint: Think in terms of number of bottles sold.)

The profit will be 0. Since we lose 5 sales for every dollar increase in price, an increase of $24 means a decrease in weekly sales of \(5 \times 24 = 120\) bottles. But we were only selling 120 bottles/week in the first place, so a decrease of 120 means we are now selling 0 bottles. If you don’t sell anything, you will make no profit. Again, students should observe that \(y(24) = 27 \cdot 0 = 0\). If either factor is zero, the product is zero.

6. You are interested in determining the price increase that will result in your maximum weekly profit. To investigate this, input the equation in your calculator as \(y_1 = (x + 3)(120 - 5x)\). Then look at your table of values and observe your weekly profit for various possible price increases. By how much will you increase the price? Why?

If the price is increased by $10.50, you will make the maximum profit. After that, further increases result in lower profit.

Students may see that 10.50 is exactly halfway between the zeros at -3 and 24 because of the symmetry of the function around its turning point. Here, students observe the pattern numerically. This arises again in question 7 when students look at the graph.

7. We will now graph the profit equation. Use WINDOW values Xmin=-5 and Xmax=30. Observe the \(y\)-values in your table to help you pick an appropriate range. Then set your values for Ymin and Ymax. Graph your function.

a. Is the graph a straight line? No. (Some students may know that the graph is a parabola.)

b. Where are the \(x\)-intercepts? \((-3,0)\) and \((24,0)\)

c. Why does this make sense? (Think about your answers to parts 4 and 5.) The values of the function represent the weekly profit when the price is increased by \(x\). We already know that when the price is reduced by $3, the profit is 0; and that when the price is increased by $24, the profit is 0. So we expect to see \(y=0\) on the graph at both those \(x\) values. When \(y=0\), the point is on the \(x\)-axis.

d. Use this graph to determine how much you will increase the price. How does the “best” price increase compare to the price increases that result in 0 profit?

As we also saw in questions 6, the profit is highest if the price is increased by $10.50. On the graph, the symmetry about \(x=10.5\) is even more apparent. Students should see that the \(x\)-value at the turning point is exactly halfway between the two zeros. \((-3+24)/2 = 10.5\).
Make sure you discuss Problem 7 and students see the connection between the x-intercepts and zero profit occurring at the roots, where you either receive 0 profit per bottle or sell 0 bottles. These occurred WHEN A LINE WAS AT y = 0.

8. Question 8 is important for showing students the standard form of a quadratic function and relating it to the factored form. Questions 6 and 7 give students an opportunity to observe the symmetry of the function. In question 8, the word is introduced.

You may want students to graph the function by hand. If so, they could use coordinate values from the calculator table. Then they can draw the mirror line, x=10.5. To do the same thing with a TI-83 calculator, you can use the Draw menu. With the graph of the quadratic already made, go to Draw from the Home screen and choose Vertical. On the Home screen, enter Vertical 10.5. The vertical line will appear on your graph in the appropriate position.

9. How does the factored form, \( y = (x + 3)(120 - 5x) \), inform us of what we will see on the graph?

An answer to question 9 might be as follows. From the factored form, we can see where the graph of the quadratic will intersect the x-axis because it is easy to tell where a factor has a value of 0. When a factor is 0, then the product will be 0, so there will be a point on the graph with a y-coordinate of 0. All the points where y = 0 are on the x-axis. Once we know where the graph crosses the x-axis, we also know that the vertex of the graph will be at an x-value half-way between those two x-values.

The following set of questions is similar to the investigation above. You can use it as homework or a quiz or perhaps as a test question. Its purpose is both to reinforce the mathematical content of the lesson above and to assure that students think carefully about the development of the linear functions used to build the quadratic function explored in this lesson.

**Traditional Brooms and Quadratic Equations**

A woman and her daughters make traditional brooms and sell them at a price that gives them a profit of $10 per broom. You have studied the market, and you tell the woman that perhaps she should change her price. You tell her that for every dollar she raises her price she will sell 2 fewer brooms each month. Right now, she sells 70 brooms each month, on average.

a. Write an expression for her profit on each broom if she raises her price for a broom by $X.

b. Write an expression for the number of brooms she will sell each month if she raises her price for a broom by $X.

c. You know that her total profit for a month depends on how many brooms she sells and what her profit on each broom is. Write an expression for her total profit in a month if she raises her price for a broom by $X.

d. What is her total profit in a month if she reduces her price by $10? Explain, using the information given.
e. What is her total profit in a month if she raises her price by $35? Explain, using the information given.

f. What is her total profit in a month if she raises her price by $10?

g. If she does not change her price, what is her profit in a month?

h. (Extra credit) What change in the price would give her the maximum monthly profit?