Why is looking at an iguana different from looking at a fish?
Pythagoras has an answer....

Light travels pretty darn fast. In fact, in air, light travels 30 cm (or about 1 foot) in a billionth of a second. One billionth of a second is called a nanosecond. Thus we can write that the speed of light is 30 cm/nanosecond or about 1 ft/nanosecond, depending on which measurement system we are using. Suppose lightning flashes 3 km away (3000m, so that's 300,000 cm).

Remembering the formula \( D = RT \) or \( T = \frac{D}{R} \), we can compute that it is only 300,000/30=10,000 nanoseconds or about one hundred-thousandths of a second before we see the flash. Sound travels a lot slower, so we wouldn't hear the thunder until nearly 10 seconds later.

Suppose you are looking at an iguana that is 6 feet below you and 8 feet from you, as in the following figure. Use the Pythagorean Theorem to compute the distance \( x \) from the iguana to your eyes. The reason we see the iguana is that light travels from the iguana to our eyes. Use the formula \( T = \frac{D}{R} \) to compute the number of nanoseconds it takes light to travel from the iguana to your eyes.

Light travels slower in water than in air, just like us. In fact, the speed of light in water is only 22.5 cm/nanosecond or 0.75 ft/nanosecond. Suppose you are swimming and you see a Parrot fish, 6 feet below you and 8 feet away, as in the diagram below. Compute how many nanoseconds it takes light to travel from the Parrot fish to your eyes. Notice that it takes several more nanoseconds than it took you to see the iguana. Big deal. We don't notice the difference of a few billionths of a second. Or do we?
Have you ever noticed that when you look into water, objects aren’t where they appear to be? One way to see this is to stick a pencil halfway into a glass of water. It will appear that the pencil is bent. Why is this? The answer is because light travels at a different speed in air and in water, and that light travels the path that takes the least amount of time.

Suppose you have a container of water that is 12 cm deep, and there is a mark on the bottom of the container. If you look at the mark from a point that is 50 cm away and 20 cm above the water level, there are many paths from the mark to your eyes. One such path is seen in the following figure. But light chooses the path that takes the least time. Use the Pythagorean Theorem to compute $w$, the distance the light traveled through the water. Use it again to compute $a$, the distance the light traveled through the air. Use these distances to compute the number of nanoseconds it took the light to travel through the water and the number of nanoseconds it took the light to travel through the air. Give each answer to 4 decimal places. How long would it take the light to travel that path from the mark to your eyes, to 4 decimal places?

Compute the length of time, to 4 decimal places, it would take light to travel the path seen in the next figure. Compare your answer to the previous answer.
Pick your own path and determine the length of time it will take light to travel that path from the mark to your eyes. Compare this answer to the other 2 answers.

Compute the time it takes light to travel the path seen in the next figure. That time will be smaller than the time it takes to travel any other possible path. Light actually travels the path that takes the least amount of time. So this is the actual path light will take.

From your perspective, the mark will seem to be further away than it actually is. In fact, it will appear to be where the fake mark is seen in the next figure.
Remark: Since light travels slower in water than in air, it tends to take a short path through water and a longer path through air. In fact, no matter where you observe the underwater mark from, the light will never take a path that exits the water 14 cm or more from above the mark. Thus, light traveling such a path will reflect back into the water. See the following figure. This means that when you are underwater looking up, you can only see outside of the water that is (nearly) directly over your head.

Teaching Guide for

Why is looking at an iguana different from looking at a fish?
Pythagoras has an answer....

Introduction: The mathematics in this investigation includes use of Pythagoras's Theorem, experience with square root expressions, the distance = rate x time relationship for constant speeds, and foundations for the concept of the minimum value of a function. Knowledge of Pythagoras's Theorem is a prerequisite.

It is a very good idea to take to class a container similar to the one in the diagrams in the lesson, with a mark on the floor of the container at one end. It is even a good idea to fill the container with water. You are not going to have students do the actual experiment, but it really is helpful for students to see the apparatus. Some students may have had the experience of reaching for something underwater and discovering it was farther away then they had thought. Such discussion is helpful. Introduce the lesson by demonstrating the container (imagining it filled with water if you have not filled it) and asking students about their experiences of viewing an object under water. Tell them that the reason we see an object under water in a different
Have students read the introduction to the lesson. If you have water in your container, they might want to observe the “bent pencil” phenomenon. Then they should proceed with the lesson working in groups of 3 or 4 students.

**Answers and teaching suggestions:**

The questions in this activity are not numbered. In this guide, the questions will appear in bold font with answers and discussion below the question.

….you are looking at an iguana that is 6 feet below you and 8 feet from you…. **Use the Pythagorean Theorem to compute the distance x from the iguana to your eyes.**

*Where x is the iguana’s distance from the eye,  \( x^2 = 6^2 + 8^2 \), so \( x = 10 \).*

... **Use the formula**  

\[
T = \frac{D}{R}
\]

... **to compute the number of nanoseconds it takes light to travel from the iguana to your eyes.**

\[
T = \frac{10 \text{ ft}}{1 \text{ ft/nanosec}} = 10 \text{ nanoseconds.}
\]

... the speed of light in water is only 22.5 cm/nanosecond or 0.75 ft/nanosecond. Suppose you are swimming and you see a Parrot fish, 6 feet below you and 8 feet away.... **Compute how many nanoseconds it takes light to travel from the Parrot fish to your eyes.**

*The parrot fish is also 10 feet away from the eye. In this case,*

\[
T = \frac{10 \text{ ft}}{.75 \text{ ft/nanosec}} = 13.33... \text{ nanosec.}
\]

**Notice that it takes several more nanoseconds than it took you to see the iguana.** Big deal. We don’t notice the difference of a few billionths of a second. Or do we? **Be sure students contemplate this difference.** **This investigation is about just how our eye does “notice” a few billionths of a second.**

...container of water that is 12 cm deep, and there is a mark on the bottom of the container. If you look at the mark from a point that is 50 cm away and 20 cm above the water level, there are many paths from the mark to your eyes. One such path is seen in the following figure. But light chooses the path that takes the least time. **Use the Pythagorean Theorem to compute w, the distance the light traveled through the water.**

\[
w = \sqrt{5^2 + 12^2} = 13 \text{ cm}
\]

**Use it again to compute a, the distance the light traveled through the air.**

\[
a = \sqrt{45^2 + 20^2} \approx 49.24429 \text{ cm}
\]

**Use these distances to compute the number of nanoseconds it took the light to travel through the water and the number of nanoseconds it took the light to travel through the**
air. Give each answer to 4 decimal places. How long would it take the light to travel that path from the mark to your eyes, to 4 decimal places?

Since we know (from the introduction) that light travels at an approximate rate of 22.5 cm/nanosecond in clear water (and we assume the water in the container is quite clear), we calculate that to travel 13 cm through water takes \( t \) nanoseconds with \( \frac{13}{22.5} \approx 0.5778 \) nanoseconds.

We know that light travels at about 30 cm/nanosecond in air, so to travel 49.24429 cm will take \( \frac{49.24429}{30} \approx 1.6415 \) nanoseconds.

Then altogether the light takes about \( 0.5778 + 1.6415 = 2.2193 \) nanoseconds to travel from the mark on the bottom of the container to your eye if it travels via the point on the surface of the water 5 cm from the end of the container as this picture shows.

Compute the length of time, to 4 decimal places, it would take light to travel the path seen in the next figure. Compare your answer to the previous answer.

In this case, \( w = \sqrt{15^2 + 12^2} \approx 19.20937 \text{ cm} \) and \( a = \sqrt{35^2 + 20^2} \approx 40.31129 \text{ cm} \).

Thus, it would take about \( \frac{19.20937}{22.5} \approx 0.85375 \) nanoseconds to travel through water and about \( \frac{40.31129}{30} \approx 1.34371 \) nanoseconds to travel through air. If light traveled via this new
position on the surface of the water, the total amount of time it would take light to travel from the object to the eye is thus 0.87375 + 1.34371 = 2.1975 nanoseconds. Compare this to the total time of 2.2193 nanoseconds when we calculated with a path via the first position. This path would take less time.

Pick your own path and determine the length of time it will take light to travel that path from the mark to your eyes. Compare this answer to the other 2 answers.

Each group should choose a different position for the possible point of exit of light from the water. Collect the data from each group in a table on the chalkboard, and include the information already calculated above so that students may see how the time decreases then increases as the exit point guesses increase in distance from the end of the container. Thus for example:

<table>
<thead>
<tr>
<th>Distance of exit point from far end of container (cm)</th>
<th>Time in water, ( \frac{w}{22.5} ) (nanoseconds)</th>
<th>Time in air, ( \frac{a}{30} ) (nanoseconds)</th>
<th>Total time (nanoseconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.5778</td>
<td>1.6415</td>
<td>2.2193</td>
</tr>
<tr>
<td>15</td>
<td>0.85375</td>
<td>1.34371</td>
<td>2.1975</td>
</tr>
</tbody>
</table>

Finally:

Compute the time it takes light to travel the path seen in the next figure. That time will be smaller than the time it takes to travel any other possible path. Light actually travels the path that takes the least amount of time. So this is the actual path light will take.

If you put the results of this into the table, the line will read:  10.8, 0.71753, 1.46691, 2.1844

![Diagram](image)

If your class does this lesson and they earlier did the lesson titled *Equations, Lenses and Fractions*, then in this one you should observe that lenses change the direction of the light because of the difference of the speed of light in air and in the lens. You should also note that in the functioning of the eye, light goes from air, through the lens, then through the liquid in the
eye, so there are 3 different speeds involved. Thus to understand the functioning of the eye, the mathematics from both of these lessons would have to be combined.

This investigation has explored the minimum value of the function

\[ T(d) = \frac{\sqrt{12^2 + d^2}}{22.5} + \frac{\sqrt{20^2 + (50 - d)^2}}{30}. \]

However, its purpose in this course is simply to build a foundation for future study of optimization while getting some experience with Pythagoras’s Theorem and the distance=rate x time relationship. In a different course, students might build the algebraic function and explore its graph. You will find a lesson appropriate for a pre-calculus class at [www.georgetown.edu/handsonmath](http://www.georgetown.edu/handsonmath) under the title “Speed of Light in Water.”
Afterword

These mathematics lessons will also soon be available on line at www.uvi.edu/..... From the website, you can download it onto your computer and make modifications that will make the lessons more relevant to your own environment and your own community of students. This can be done with a minimum of research to get data that will be of direct concern to your students. You can also update information in the lessons. We request that you e-mail rdance@uvi.edu to request permission for downloads each time you do one. Permission will be cheerfully granted, but we need to keep track of use.

Very best wishes for productive use of these materials. Please let us know about your experience in using them.

Rosalie A. Dance, Mathematics Professor
University of the Virgin Islands
St. Thomas, USVI
00802
rdance@uvi.edu

James T. Sandefur, Mathematics Professor
Georgetown University
Washington, DC, USA
20057
sandefur@georgetown.edu

The development of these materials was supported, in part, by the National Science Foundation. Opinions expressed herein are those of the authors and not necessarily those of the Foundation. Thanks for support are due to NSF, to Dr. Lee Zia, and to those who provided information about contexts, including

Professor Gene Emmanuel, University of the Virgin Islands
Dr. Barbara Kojis, Virgin Islands Fish and Wildlife Service
Dr. Frank Rinehart, Monroe Community College (formerly of UVI)
Professor Roy Watlington, University of the Virgin Islands